Pointwise multipliers on martingale Campanato spaces

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Abstract

We introduce generalized Campanato spaces $\mathcal{L}_{p,\phi}$ on a probability space $(\Omega, \mathcal{F}, P)$, where $p \in [1, \infty)$ and $\phi : (0, 1] \to (0, \infty)$. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{p,\phi} = \mathrm{BMO}$. We give a characterization of the set of all pointwise multipliers on $\mathcal{L}_{p,\phi}$.

1 Introduction

We consider a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$, where $\{\mathcal{F}_n\}_{n\geq 0}$ is a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. For the sake of simplicity, let $\mathcal{F}_{-1} = \mathcal{F}_0$. We suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, where $B \in \mathcal{F}_n$ is called an atom (more precisely a $(\mathcal{F}_n, P)$-atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively.

Let $\mathcal{X}$ be a normed space of $\mathcal{F}$-measurable functions. We say that an $\mathcal{F}$-measurable function g is a pointwise multiplier on $\mathcal{X}$, if the pointwise multiplication fg is in $\mathcal{X}$ for any $f \in \mathcal{X}$. We denote by $\mathrm{PWM}(\mathcal{X})$ the set of all pointwise multipliers on $\mathcal{X}$. If $\mathcal{X}$ is a Banach space and has the following

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property, then every $g \in \mathrm{PWM}(\mathcal{X})$ is a bounded operator on $\mathcal{X}$.

$$(1.1) \quad f_n \to f \text{ in } \mathcal{X} \ (n \to \infty) \implies \exists \{n(j)\} \text{ s.t. } f_{n(j)} \to f \text{ a.s. } (j \to \infty).$$

Actually, from (1.1) we see that g is a closed operator. Therefore, g is a bounded operator by the closed graph theorem.

It is known that $\mathrm{PWM}(L_p) = L_\infty$ for $p \in (0, \infty]$. More generally, if $\mathcal{X}$ is a (quasi) Banach function space, then $\mathrm{PWM}(\mathcal{X}) = L_\infty$ (see [4, 7]). For Banach function spaces, see Kikuchi [2].

In this paper we consider the pointwise multipliers on generalized Campanato spaces which are not Banach function spaces in general. We always assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, that is, the operator E_0 coincides with E. Then we introduce generalized Campanato spaces $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^\natural_{p,\phi}$ as the following:

Definition 1.1. Let $p \in [1, \infty)$ and ϕ be a function from $(0, 1]$ to $(0, \infty)$. For $f \in L_1$, let

$$(1.2) \qquad \|f\|_{\mathcal{L}_{p,\phi}} = \sup_{n \geq 0} \sup_{B \in A(\mathcal{F}_n)} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B |f - E_n f|^p \, dP \right)^{1/p},$$

and

$$(1.3) \qquad \|f\|_{\mathcal{L}^\natural_{p,\phi}} = \|f\|_{\mathcal{L}_{p,\phi}} + |Ef|.$$

Define

$$\mathcal{L}_{p,\phi} = \{f \in L_1 : \|f\|_{\mathcal{L}_{p,\phi}} < \infty\} \quad \text{and} \quad \mathcal{L}^\natural_{p,\phi} = \{f \in L_1 : \|f\|_{\mathcal{L}^\natural_{p,\phi}} < \infty\}.$$

If $\phi(r) = r^\lambda$, $\lambda \in (-\infty, \infty)$, we simply denote $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^\natural_{p,\phi}$ by $\mathcal{L}_{p,\lambda}$ and $\mathcal{L}^\natural_{p,\lambda}$, respectively, which introduced by [9].

Note that $\mathcal{L}_{p,\phi}$ and $\mathcal{L}^\natural_{p,\phi}$ are coincide as sets of measurable functions. We regard $\mathcal{L}_{p,\phi} = (\mathcal{L}_{p,\phi}, \|\cdot\|_{\mathcal{L}_{p,\phi}})$ is a seminormed space and $\mathcal{L}^\natural_{p,\phi} = (\mathcal{L}^\natural_{p,\phi}, \|\cdot\|_{\mathcal{L}^\natural_{p,\phi}})$ is a normed space. Then $\mathcal{L}^\natural_{p,\phi}$ is a Banach space, but it is not a Banach function space in general. It is easy to see that $\mathcal{L}^\natural_{p,\phi}$ has the property (1.1), since

$$\|f\|_{L_1} \leq E[|f - Ef|] + |Ef| \leq \max(1, \phi(1)) \|f\|_{\mathcal{L}^\natural_{p,\phi}}.$$

For $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$, let

$$\|g\|_{Op} = \sup_{f \neq 0} \frac{\|fg\|_{\mathcal{L}^\natural_{p,\phi}}}{\|f\|_{\mathcal{L}^\natural_{p,\phi}}}.$$

We also define BMO and Lip_α as the following:

Definition 1.2. For $\phi \equiv 1$, denote $\mathcal{L}_{1,\phi}$ and $\mathcal{L}_{1,\phi}^{\natural}$ by BMO and $\mathrm{BMO}^{\natural}$, respectively. For $\phi(r) = r^{\alpha}$, $\alpha > 0$, denote $\mathcal{L}_{1,\phi}$ and $\mathcal{L}_{1,\phi}^{\natural}$ by Lip_{α} and $\mathrm{Lip}_{\alpha}^{\natural}$, respectively.

Let

$$L_{1,0} = \{f \in L_1 : Ef = 0\}.$$

Then $\mathrm{BMO} \cap L_{1,0} = \mathrm{BMO}^{\natural} \cap L_{1,0}$ and $\mathrm{Lip}_{\alpha} \cap L_{1,0} = \mathrm{Lip}_{\alpha}^{\natural} \cap L_{1,0}$. These spaces coincide with BMO and Lip_{α} defined by Weisz [12], respectively, under the assumption that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, see [9] for details.

We say $\{\mathcal{F}_n\}_{n\geq 0}$ is regular if there exists $R \geq 2$ such that

$$f_n \leq R f_{n-1} \text{ for all non-negative martingale } f = (f_n)_{n\geq 0}. \tag{1.4}$$

A function $\theta : (0,1] \to (0,\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$\frac{1}{C} \leq \frac{\theta(r)}{\theta(s)} \leq C \quad \text{for} \quad r, s \in (0,1],\ \frac{1}{2} \leq \frac{r}{s} \leq 2.$$

A function $\theta : (0,1] \to (0,\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\theta(r) \leq C\theta(s) \quad (\theta(r) \geq C\theta(s)) \quad \text{for} \quad 0 < r \leq s \leq 1.$$

Our main result is the following:

Theorem 1.1. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition and that*

$$\int_0^r \phi(t)^p \, dt \leq C r \phi(r)^p \quad \text{for all } r \in (0,1]. \tag{1.5}$$

Let

$$\phi_*(r) = 1 + \int_r^1 \frac{\phi(t)}{t} \, dt. \tag{1.6}$$

Then

$$\mathrm{PWM}(\mathcal{L}_{p,\phi}^{\natural}) = \mathcal{L}_{p,\phi/\phi_*} \cap L_\infty.$$

Moreover, for $g \in \mathrm{PWM}(\mathcal{L}_{p,\phi}^{\natural})$, $\|g\|_{Op}$ is equivalent to $\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \|g\|_{L_\infty}$.

See [1, 6, 10, 11] for pointwise multipliers on BMO and Campanato spaces defined on the Euclidean space. Our basic idea comes from [1, 10].

Remark 1.1. (i) If ϕ satisfies the doubling condition and (1.5), then $r\phi(r)^p$ is almost increasing.

(ii) If ϕ is almost increasing, then ϕ/ϕ_* is also.

(iii) Let

$$\|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}} = \sup_{n\ge 0} \sup_{A\in\mathcal{F}_n} \frac{1}{\phi(P(A))} \left(\frac{1}{P(A)} \int_A |f-E_nf|^p \, dP\right)^{1/p}. \tag{1.7}$$

Then $\|f\|_{\mathcal{L}_{p,\phi}} \le \|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}}$ by the definition. If ϕ is almost increasing, then $\|f\|_{\mathcal{L}_{p,\phi}}$ and $\|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}}$ are equivalent. Actually, for any $A \in \mathcal{F}_n$, there exists a sequence of atoms $B_\ell \in A(\mathcal{F}_n)$, $\ell = 1, 2, \cdots$, such that $A = \cup_\ell B_\ell$ and $P(A) = \sum_\ell P(B_\ell)$. Then

$$\begin{aligned}\int_A |f-E_nf|^p \, dP &= \sum_\ell \int_{B_\ell} |f-E_nf|^p \, dP \\ &\le \sum_\ell \phi(P(B_\ell))^p P(B_\ell) \|f\|_{\mathcal{L}_{p,\phi}}^p \\ &\le C^p \phi(P(A))^p P(A) \|f\|_{\mathcal{L}_{p,\phi}}^p.\end{aligned}$$

This shows $\|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}} \le C\|f\|_{\mathcal{L}_{p,\phi}}$. If ϕ is not almost increasing, then $\|f\|_{\mathcal{L}_{p,\phi}}$ is not equivalent to $\|f\|_{\mathcal{L}_{p,\phi,\mathcal{F}}}$ in general, see [9]. The norm (1.7) was introduced by [5] for general $\{\mathcal{F}_n\}_{n\ge 0}$.

By Theorem 1.1 we have the next two corollaries immediately:

Corollary 1.2. *Let $\{\mathcal{F}_n\}_{n\ge 0}$ be regular and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Then*

$$\mathrm{PWM}(\mathrm{BMO}^\natural) = \mathcal{L}_{1,\psi} \cap L_\infty,$$

where $\psi(r) = 1/\log(e/r)$. Moreover, for $g \in \mathrm{PWM}(\mathrm{BMO}^\natural)$, $\|g\|_{Op}$ is equivalent to $\|g\|_{\mathcal{L}_{1,\psi}} + \|g\|_{L_\infty}$.

Corollary 1.3. *Let $\{\mathcal{F}_n\}_{n\ge 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\alpha > 0$. Then*

$$\mathrm{PWM}(\mathrm{Lip}_\alpha^\natural) = \mathrm{Lip}_\alpha \cap L_\infty.$$

Moreover, for $g \in \mathrm{PWM}(\mathrm{Lip}_\alpha^\natural)$, $\|g\|_{Op}$ is equivalent to $\|g\|_{\mathrm{Lip}_\alpha} + \|g\|_{L_\infty}$.

Example 1.1. Let $\{\mathcal{F}_n\}_{n\geq 0}$, p and ϕ satisfy the assumption in Theorem 1.1. For a sequence

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathcal{F}_n),$$

let

$$g = \sin h, \quad \text{where} \quad h = \sum_{n=1}^{\infty} \frac{\phi(P(B_n))}{\phi_*(P(B_n))} \left(\frac{P(B_{n-1})}{P(B_n)} \chi_{B_n} - \chi_{B_{n-1}} \right). \tag{1.8}$$

Then h is in $\mathcal{L}_{p,\phi/\phi_*}$, see Lemma 2.4 and Remarks 2.1. Hence $g \in \mathcal{L}_{p,\phi/\phi_*} \cap L_\infty$, since $\sin\theta$ is Lipschitz continuous, see Remark 2.3. That is, $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$. If $\phi \equiv 1$, then $\phi(r)/\phi_*(r) = 1/\log(e/r)$ and $g \in \mathrm{PWM}(\mathrm{BMO}^\natural)$.

Next, for a martingale $(f_n)_{n\geq 0}$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$, it is said to be $\mathcal{L}_{p,\lambda}$-bounded if $f_n \in \mathcal{L}_{p,\lambda}$ $(n \geq 0)$ and $\sup_{n\geq 0} \|f_n\|_{\mathcal{L}_{p,\lambda}} < \infty$. Similarly, the martingale $(f_n)_{n\geq 0}$ is said to be $\mathcal{L}^\natural_{p,\lambda}$-bounded if $f_n \in \mathcal{L}^\natural_{p,\lambda}$ $(n \geq 0)$ and $\sup_{n\geq 0} \|f_n\|_{\mathcal{L}^\natural_{p,\lambda}} < \infty$.

Let

$$\mathcal{L}_{p,\phi}(\mathcal{F}_n) = \{f \in L_1 : f \text{ is } \mathcal{F}_n\text{-measurable and } \|f\|_{\mathcal{L}_{p,\phi}} < \infty\}$$

and

$$\mathcal{L}^\natural_{p,\phi}(\mathcal{F}_n) = \{f \in L_1 : f \text{ is } \mathcal{F}_n\text{-measurable and } \|f\|_{\mathcal{L}^\natural_{p,\phi}} < \infty\}.$$

Then we have the following:

Theorem 1.4. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition and* (1.5)*. Let $g \in L_1$ and $(g_n)_{n\geq 0}$ be its corresponding martingale with $g_n = E_n g$ $(n \geq 0)$. If $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$, then $g_n \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}(\mathcal{F}_n))$. Conversely, if $g_n \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}(\mathcal{F}_n))$ and $\sup_{n\geq 0} \|g_n\|_{Op} < \infty$, then $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$.*

We show several lemmas in Section 2 to prove Theorem 1.1 in Section 3. We prove Theorem 1.4 in Section 4.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, is dependent on the subscripts. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2 Lemmas

To prove Theorem 1.1 we show several lemmas in this section. The first lemma was proved in [9].

Lemma 2.1 ([9, Lemma 3.3])**.** *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular. Then every sequence*

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathcal{F}_n)$$

has the following property; for each $n \geq 1$,

$$B_n = B_{n-1} \text{ or } \left(1 + \frac{1}{R}\right) P(B_n) \leq P(B_{n-1}) \leq RP(B_n),$$

where R is the constant in (1.4).

For a function $f \in L_1$ and an atom $B \in A(\mathcal{F}_n)$, let

$$f_B = \frac{1}{P(B)} \int_B f \, dP.$$

For a function $\phi : (0,1] \to (0,\infty)$, let ϕ_* be defined by (1.6). If ϕ satisfies the doubling condition, then $\phi(r) \leq C\phi_*(r)$ for all $r \in (0,1]$.

Lemma 2.2. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition. For $f \in \mathcal{L}^\natural_{p,\phi}$ and $B \in \cup_{n\geq 0} A(\mathcal{F}_n)$,*

$$|f_B| \leq C\phi_*(P(B)) \|f\|_{\mathcal{L}^\natural_{p,\phi}}. \tag{2.1}$$

Proof. By Lemma 2.1, we can choose $B_{k_j} \in A(\mathcal{F}_{k_j})$, $0 = k_0 < k_1 < \cdots < k_m \leq n$, such that $B_{k_0} \supset B_{k_1} \supset B_{k_2} \supset \cdots \supset B_{k_m} = B$ and that $(1 + 1/R)P(B_{k_j}) \leq P(B_{k_{j-1}}) \leq RP(B_{k_j})$. Then, we have

$$\begin{aligned}
|f_{B_{k_j}} - f_{B_{k_{j-1}}}| &= \left| \frac{1}{P(B_{k_j})} \int_{B_{k_j}} f(\omega)\, dP - \frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}}} f(\omega)\, dP \right| \\
&= \left| \frac{1}{P(B_{k_j})} \int_{B_{k_j}} [f - E_{k_{j-1}} f](\omega)\, dP \right| \\
&\leq \left(\frac{1}{P(B_{k_j})} \int_{B_{k_j}} |f - E_{k_{j-1}} f|^p \, dP \right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\frac{1}{P(B_{k_{j-1}})} \int_{B_{k_{j-1}}} |f - E_{k_{j-1}} f|^p \, dP \right)^{1/p} \\
&\leq \phi(P(B_{k_{j-1}})) \, \|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.
\end{aligned}
$$

Since ϕ satisfies the doubling condition,

$$
\begin{aligned}
|f_B - f_{B_0}| &\leq \sum_{j=1}^{m} |f_{B_{k_j}} - f_{B_{k_{j-1}}}| \\
&\lesssim \sum_{j=1}^{m} \phi(P(B_{k_{j-1}})) \, \|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \\
&\lesssim \sum_{j=1}^{m} \int_{P(B_{k_j})}^{P(B_{k_{j-1}})} \frac{\phi(t)}{t} \, dt \, \|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \\
&= \int_{P(B)}^{1} \frac{\phi(t)}{t} \, dt \, \|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \\
&= \{\phi_*(P(B)) - 1\} \, \|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.
\end{aligned}
$$

On the other hand,

$$
|f_{B_0}| = |Ef| \leq \|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.
$$

Therefore, we have (2.1). $\square$

Lemma 2.3. *Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1, \infty)$ and $\phi : (0, 1] \to (0, \infty)$. Assume that $r\phi(r)^p$ is almost increasing. For any atom $B \in \cup_{n \geq 0} A(\mathcal{F}_n)$, the characteristic function χ_B is in $\mathcal{L}^{\natural}_{p,\phi}$ and there exists a positive constant C, independent of B, such that*

$$
\|\chi_B\|_{\mathcal{L}^{\natural}_{p,\phi}} \leq \frac{C}{\phi(P(B))}. \tag{2.2}
$$

Proof. Let $B \in A(\mathcal{F}_n)$ and $B' \in A(\mathcal{F}_k)$. Let $B_j \in A(\mathcal{F}_j)$, $0 \leq j \leq n$, such that $B_0 \supset B_1 \supset \cdots \supset B_n = B$.

If $k \geq n$, then $\chi_B - E_k \chi_B = 0$ and

$$
\int_{B'} |\chi_B - E_k \chi_B|^p \, dP = 0.
$$

If $k < n$ and $B' \neq B_k$, then $B' \cap B_k = \emptyset$ and

$$
\int_{B'} |\chi_B - E_k \chi_B|^p \, dP = 0.
$$

Hence, we have

$$\|\chi_B\|_{\mathcal{L}_{p,\phi}} = \sup_{k<n} \frac{1}{\phi(P(B_k))} \left(\frac{1}{P(B_k)} \int_{B_k} |\chi_B - E_k \chi_B|^p \, dP \right)^{1/p}.$$

For $k < n$, since $r\phi(r)^p$ is almost increasing,

$$\begin{aligned}
&\frac{1}{\phi(P(B_k))^p} \frac{1}{P(B_k)} \int_{B_k} |\chi_B - E_k \chi_B|^p \, dP \\
&= \frac{1}{\phi(P(B_k))^p P(B_k)} \left\{ P(B_n) \left(1 - \frac{P(B_n)}{P(B_k)}\right)^p + (P(B_k) - P(B_n)) \left(\frac{P(B_n)}{P(B_k)}\right)^p \right\} \\
&\lesssim \frac{1}{\phi(P(B_n))^p P(B_n)} \left\{ P(B_n) \left(1 - \frac{P(B_n)}{P(B_k)}\right)^p + (P(B_k) - P(B_n)) \left(\frac{P(B_n)}{P(B_k)}\right)^p \right\} \\
&= \frac{1}{\phi(P(B_n))^p} \left\{ \left(1 - \frac{P(B_n)}{P(B_k)}\right)^p + \left(1 - \frac{P(B_n)}{P(B_k)}\right) \left(\frac{P(B_n)}{P(B_k)}\right)^{p-1} \right\} \\
&\lesssim \frac{1}{\phi(P(B_n))^p} = \frac{1}{\phi(P(B))^p}.
\end{aligned}$$

Therefore, we have

$$\|\chi_B\|_{\mathcal{L}_{p,\phi}} \lesssim \frac{1}{\phi(P(B))}. \tag{2.3}$$

On the other hand, since $r\phi(r)^p$ is almost increasing,

$$|E\chi_B| = P(B) \le P(B)^{1/p} \lesssim \frac{1}{\phi(P(B))}. \tag{2.4}$$

Combining (2.3) and (2.4), we have (2.2). □

Lemma 2.4. *Let $\{\mathcal{F}_n\}_{n\ge 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ satisfies the doubling condition and (1.5). For a sequence*

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots, \quad B_n \in A(\mathcal{F}_n),$$

let

$$f_0 = \chi_{B_0}, \quad u_k = \phi(P(B_k)) \left(\frac{P(B_{k-1})}{P(B_k)} \chi_{B_k} - \chi_{B_{k-1}} \right),$$

and let

$$f_n = f_0 + \sum_{k=1}^{n} u_k. \tag{2.5}$$

Then $(f_n)_{n\ge 0}$ is a martingale and $\mathcal{L}^{\natural}_{p,\phi}$-bounded. The sum $f \equiv f_0 + \sum_{k=1}^{\infty} u_k$ converges a.s. and in L_p, and $E_n f = f_n$ for $n \ge 0$. Moreover, there exist positive constants C_1 and C_2, independent of the sequence of atoms, such that

$$\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \le C_1 \quad \textit{and} \quad |f_{B_n}| \ge C_2\phi_*(P(B_n)), \; n \ge 0. \tag{2.6}$$

Proof. Since $E_n[u_k] = 0$ for $k > n$, $(f_n)_{n\ge 0}$ is a martingale. We show that the sum $f_0 + \sum_{k=1}^{\infty} u_k$ converges in L_p. If $\lim_{k\to\infty} P(B_k) > 0$ then the convergence is clear because there exists m such that $B_m = B_n$ for all $n \ge m$. We assume that $\lim_{k\to\infty} P(B_k) = 0$. By Lemma 2.1, we can take a sequence of integers $0 = k_0 < k_1 < \cdots < k_j < \cdots$ that satisfies

$$(1 + 1/R)P(B_{k_j}) \le P(B_{k_{j-1}}) \le RP(B_{k_j}), \tag{2.7}$$

and $B_{k_{j-1}} = B_k$ if $k_{j-1} \le k < k_j$. In this case we can write

$$f_n = \chi_{B_0} + \sum_{1\le k_j \le n} \phi(P(B_{k_j})) \left(\frac{P(B_{k_{j-1})}}{P(B_{k_j})} \chi_{B_{k_j}} - \chi_{B_{k_{j-1}}} \right).$$

Note that, by Remark 1.1 and [8, Lemma 7.1], the doubling condition and (1.5) implies

$$\int_0^r \phi(t) t^{1/p-1}\, dt \le C_p \phi(r) r^{1/p} \quad \text{for all } r \in (0, 1]. \tag{2.8}$$

Using the doubling condition and (2.8), we have

$$\begin{aligned}
&\sum_{k_j > n} \phi(P(B_{k_j})) \left\| \frac{P(B_{k_{j-1})}}{P(B_{k_j})} \chi_{B_{k_j}} - \chi_{B_{k_{j-1}}} \right\|_{L_p} \\
&\le \sum_{k_j>n} \phi(P(B_{k_j}))(R\|\chi_{B_{k_j}}\|_{L_p} + \|\chi_{B_{k_{j-1}}}\|_{L_p}) \\
&\le 2R \sum_{k_j>n} \phi(P(B_{k_j}))P(B_{k_j})^{1/p} \\
&\le C \sum_{k_j>n} \int_{P(B_{k_j})}^{P(B_{k_{j-1}})} \phi(t) t^{1/p-1}\, dt \\
&\le C \int_0^{P(B_n)} \phi(t) t^{1/p-1}\, dt \\
&\le CC_p \phi(P(B_n))P(B_n)^{1/p}.
\end{aligned} \tag{2.9}$$

We can deduce from (2.9) that $f \equiv f_0 + \sum_{k=1}^{\infty} u_k$ converges in L_p. By the martingale convergence theorem, $f_0 + \sum_{k=1}^{\infty} u_k$ also converges almost surely. Moreover, we have $E_n f = f_n$ and

$$\left(\frac{1}{P(B_n)} \int_{B_n} |f - E_n f|^p \, dP \right)^{1/p} \leq C C_p \phi(P(B_n)). \tag{2.10}$$

For $B' \in A(\mathcal{F}_n)$, we have

$$(f - E_n f)\chi_{B'} = \begin{cases} f - E_n f & (B' = B_n) \\ 0 & (B' \neq B_n). \end{cases} \tag{2.11}$$

Combining (2.10) and (2.11), we have $\|f\|_{\mathcal{L}_{p,\phi}} \leq C$ where C is a positive constant independent of the sequence of atoms. Moreover, since $B_0 = \Omega$,

$$|Ef| = |f_0| = 1.$$

Therefore, $\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \leq C_1$ where C_1 is a positive constant independent of the sequence of atoms.

We now show $|f_{B_n}| \geq C_2 \phi_*(P(B_n))$. On the atom B_n, we have

$$f_n = 1 + \sum_{1 \leq k_j \leq n} \phi(P(B_{k_j})) \left(\frac{P(B_{k_{j-1}})}{P(B_{k_j})} - 1 \right) \geq 1 + \frac{1}{R} \sum_{1 \leq k_j \leq n} \phi(P(B_{k_j})).$$

Therefore, we have

$$\begin{aligned} |f_{B_n}| &= \left| \frac{1}{P(B_n)} \int_{B_n} f_n \, dP \right| \\ &\geq 1 + \frac{1}{R} \sum_{1 \leq k_j \leq n} \phi(P(B_{k_j})) \\ &\sim 1 + \sum_{1 \leq k_j \leq n} \int_{P(B_{k_j})}^{P(B_{k_{j-1}})} \frac{\phi(t)}{t} \, dt \\ &= 1 + \int_{P(B_n)}^{1} \frac{\phi(t)}{t} \, dt = \phi_*(P(B_n)) \end{aligned}$$

That is, $|f_{B_n}| \geq C_2 \phi_*(P(B_n))$ where C_2 is a positive constant independent of the sequence of atoms. □

Remark 2.1. From the proof on Lemma 2.4 we see that, for

$$h = \sum_{k=1}^{\infty} u_k, \quad h_0 = 0, \quad h_n = \sum_{k=1}^{n} u_k \ (n \geq 1), \tag{2.12}$$

h is in $\mathcal{L}_{p,\phi}$ and $(h_n)_{n\geq 0}$ is its corresponding martingale with $h_n = E_n h$ $(n \geq 0)$.

Remark 2.2. Let $(\Omega, \mathcal{F}, P)$ be as follows:

$$\Omega = [0,1), \quad A(\mathcal{F}_n) = \{I_{n,j} = [j2^{-n}, (j+1)2^{-n}) : j = 0, 1, \cdots, 2^n - 1\}$$
$$\mathcal{F}_n = \sigma(A(\mathcal{F}_n)), \quad \mathcal{F} = \sigma(\cup_n \mathcal{F}_n), \quad P = \text{ the Lebesgue measure.}$$

If $\phi(r) = 1/\log(e/r)$, then h in (2.12) is unbounded. Actually,

$$u_k = \frac{1}{1 + k\log 2}(2\chi_{B_k} - \chi_{B_{k-1}}),$$

and

$$h = \sum_{k=1}^{n} \frac{1}{1 + k\log 2} \quad \text{on } B_n \setminus B_{n+1}.$$

Remark 2.3. If $F : \mathbb{C} \to \mathbb{C}$ is Lipschitz continuous, that is,

$$|F(z_1) - F(z_2)| \leq C|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{C},$$

then, for $B \in \mathcal{F}_n$,

$$\int_B |F(f) - E_n[F(f)]|\, dP \leq 2C \int_B |f - E_n f|\, dP.$$

Actually,

$$\begin{aligned}
&\int_B |F(f) - E_n[F(f)]|\, dP \\
&\leq \int_B |F(f) - F(E_n f)|\, dP + \int_B |F(E_n f) - E_n[F(f)]|\, dP \\
&= \int_B |F(f) - F(E_n f)|\, dP + \int_B |E_n[F(E_n f) - F(f)]|\, dP \\
&\leq 2 \int_B |F(f) - F(E_n f)|\, dP \\
&\leq 2C \int_B |f - E_n f|\, dP.
\end{aligned}$$

Lemma 2.5. *Let $p \in [1, \infty)$ and $\phi : (0,1] \to (0, \infty)$. Suppose that $f \in \mathcal{L}_{p,\phi}$ and $g \in L_\infty$. Then $fg \in \mathcal{L}_{p,\phi}$ if and only if*

$$F(f,g) \equiv \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{|f_B|}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B |g - E_n g|^p\, dP \right)^{1/p} < \infty. \tag{2.13}$$

In this case,

$$\big| F(f,g) - \|fg\|_{\mathcal{L}_{p,\phi}} \big| \leq 2\|f\|_{\mathcal{L}_{p,\phi}} \|g\|_{L_\infty}. \tag{2.14}$$

Proof. Let $f \in \mathcal{L}_{p,\phi}$ and $g \in L_\infty$. Let $B \in A(\mathcal{F}_n)$. Since $E_n f = f_B$ on B, we can use the same method as in [6, Lemma 3.5] and we have

$$\begin{aligned}
&\left| \left(\frac{1}{P(B)} \int_B |fg - E_n[fg]|^p \, dP \right)^{1/p} - |f_B| \left(\frac{1}{P(B)} \int_B |g - E_n g|^p \, dP \right)^{1/p} \right| \\
&\le 2 \left(\frac{1}{P(B)} \int_B |(f - E_n f) g|^p \, dP \right)^{1/p} \le 2\phi(P(B)) \|f\|_{\mathcal{L}_{p,\phi}} \|g\|_{L_\infty}.
\end{aligned} \tag{2.15}$$

Therefore, $fg \in \mathcal{L}_{p,\phi}$ if and only if $F(f,g) < \infty$. In this case, we can deduce (2.14) from (2.15). □

Lemma 2.6. *Let $\{\mathcal{F}_n\}_{n\ge 0}$ be regular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that $r\phi(r)^p$ is almost increasing. If $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$, then $g \in L_\infty$ and $\|g\|_{L_\infty} \le C\|g\|_{Op}$ for some positive constant C independent of g.*

Proof. Let $g \in \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi})$. Since the constant function 1 is in $\mathcal{L}^\natural_{p,\phi}$, the pointwise multiplication $g = g \cdot 1$ is in $\mathcal{L}^\natural_{p,\phi}$, which implies $g \in L_1$. Then

$$E[|g|] \le E[|g - Eg|] + |Eg| \le \max(1, \phi(1)) \|g\|_{\mathcal{L}^\natural_{p,\phi}} \lesssim \|g\|_{Op} \|1\|_{\mathcal{L}^\natural_{p,\phi}} = \|g\|_{Op}.$$

Since $\{\mathcal{F}_n\}_{n\ge 0}$ is regular, we also have $E_n g \in L_\infty$ as follows:

$$E_n[|g|] \le R E_{n-1}[|g|] \le \cdots \le R^n E_0[|g|] = R^n E[|g|].$$

Next we shall show that there exists a positive constant C such that $\|g\|_{L_\infty} \le C\|g\|_{Op}$. Then we have the conclusion. Let $B \in A(\mathcal{F}_n)$ such that $|g_B| \ge \|E_n g\|_{L_\infty}/2$. By Lemma 2.1 there exists $B' \in A(\mathcal{F}_{n'})$ with $B \subset B'$ such that $(1 + 1/R)P(B) \le P(B') \le RP(B)$. Then, we have

$$\begin{aligned}
\|g\|_{Op} \|\chi_B\|_{\mathcal{L}^\natural_{p,\phi}} &\ge \|g\chi_B\|_{\mathcal{L}^\natural_{p,\phi}} \\
&\ge \frac{1}{\phi(P(B'))} \left(\frac{1}{P(B')} \int_{B'} |g\chi_B - E_{n'}[g\chi_B]|^p \, dP \right)^{1/p} \\
&\ge \frac{1}{\phi(P(B'))} \left(\frac{1}{P(B')} \int_{B' \setminus B} |g\chi_B - E_{n'}[g\chi_B]|^p \, dP \right)^{1/p} \\
&= \frac{1}{\phi(P(B'))} \left(\frac{1}{P(B')} \int_{B' \setminus B} |E_{n'}[[E_n g]\chi_B]|^p \, dP \right)^{1/p}.
\end{aligned}$$

Since $|[E_n g]\chi_B| = |g_B \chi_B| \geq \|E_n g\|_{L_\infty}\chi_B/2$, we have

$$\int_{B'\setminus B} |E_{n'}[[E_n g]\chi_B]|^p \, dP \geq \left(\frac{\|E_n g\|_{L_\infty}}{2}\right)^p \left(\frac{P(B)}{P(B')}\right)^p P(B' \setminus B).$$

Hence, we have

$$\|g\|_{Op}\|\chi_B\|_{\mathcal{L}^\natural_{p,\phi}} \geq \frac{\|E_n g\|_{L_\infty}}{2R(R+1)^{1/p}\phi(P(B'))}. \tag{2.16}$$

Combining (2.16) and Lemma 2.3, we have

$$\begin{aligned}
\|E_n g\|_{L_\infty} &\leq 2R(R+1)^{1/p}\phi(P(B'))\|g\|_{Op}\|\chi_B\|_{\mathcal{L}^\natural_{p,\phi}} \\
&\lesssim \|g\|_{Op}\frac{\phi(P(B'))}{\phi(P(B))} \\
&= \|g\|_{Op}\frac{P(B)^{1/p}}{P(B')^{1/p}}\frac{P(B')^{1/p}\phi(P(B'))}{P(B)^{1/p}\phi(P(B))} \\
&\lesssim \|g\|_{Op}.
\end{aligned}$$

Therefore,

$$\|g\|_{L_\infty} = \sup_{n\geq 0}\|E_n g\|_{L_\infty} \leq C\|g\|_{Op}.$$

This shows the conclusion. □

3 Proof of Theorem 1.1

We first show that

$$\mathcal{L}_{p,\phi/\phi_*} \cap L_\infty \subset \mathrm{PWM}(\mathcal{L}^\natural_{p,\phi}) \quad \text{and} \quad \|g\|_{Op} \leq C(\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \|g\|_{L_\infty}). \tag{3.1}$$

Let $g \in \mathcal{L}_{p,\phi/\phi_*} \cap L_\infty$ and $f \in \mathcal{L}^\natural_{p,\phi}$. Let $F(f,g)$ be as in Lemma 2.5. Then, by the definition of $F(f,g)$ and Lemma 2.2 we have

$$F(f,g) \leq C\|f\|_{\mathcal{L}^\natural_{p,\phi}}\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} < \infty.$$

Therefore, by Lemma 2.5, we have $fg \in \mathcal{L}_{p,\phi}$ and

$$\|fg\|_{\mathcal{L}_{p,\phi}} \leq C\|f\|_{\mathcal{L}^\natural_{p,\phi}}\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + 2\|f\|_{\mathcal{L}_{p,\phi}}\|g\|_{L_\infty}. \tag{3.2}$$

On the other hand, we have

$$|E[fg]| \leq \|g\|_{L_\infty}E[|f|] \leq \|g\|_{L_\infty}\max(1,\phi(1))\|f\|_{\mathcal{L}^\natural_{p,\phi}}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain (3.1).

We now show the converse, that is,

$$\text{(3.4)} \quad \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}) \subset \mathcal{L}_{p,\phi/\phi_*} \cap L_\infty \quad \text{and} \quad \|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \|g\|_{L_\infty} \le C\|g\|_{Op}.$$

Let $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi})$. By Lemma 2.6, we have $g \in L_\infty$ and $\|g\|_{L_\infty} \le C\|g\|_{Op}$. Let $B \in A(\mathcal{F}_n)$. We take $B_j \in A(\mathcal{F}_j)$ with $B_n = B$ such that

$$B_0 \supset B_1 \supset \cdots \supset B_n \supset \cdots .$$

Let f be the function described in Lemma 2.4. Then, combining Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}
&\frac{C_2\phi_*(P(B))}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |g - E_n g|^p \, dP\right)^{1/p} \\
&\le \frac{|f_B|}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |g - E_n g|^p \, dP\right)^{1/p} \\
&\le F(f, g) \\
&\le \|fg\|_{\mathcal{L}_{p,\phi}} + 2\|g\|_{L_\infty}\|f\|_{\mathcal{L}_{p,\phi}} \\
&\le \|g\|_{Op}\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} + 2C\|g\|_{Op}\|f\|_{\mathcal{L}_{p,\phi}} \\
&\lesssim \|g\|_{Op}\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \le C_1\|g\|_{Op}.
\end{aligned}$$

Therefore, we have (3.4).

4 Proof of Theorem 1.4

To prove Theorem 1.4 we use the following proposition. It can be shown by the same way as [9, Proposition 2.2] which deals with the case $\phi(r) = r^\lambda$, $\lambda \in (-\infty, \infty)$.

Proposition 4.1. *Let $1 \le p < \infty$ and $\phi : (0, 1] \to (0, \infty)$. Let $f \in L_1$ and $(f_n)_{n\ge0}$ be its corresponding martingale with $f_n = E_n f$ $(n \ge 0)$.*

(i) *If $f \in \mathcal{L}_{p,\phi}$, then $(f_n)_{n\ge0}$ is $\mathcal{L}_{p,\phi}$-bounded and*

$$\|f\|_{\mathcal{L}_{p,\phi}} \ge \sup_{n\ge0} \|f_n\|_{\mathcal{L}_{p,\phi}}.$$

Conversely, if $(f_n)_{n\ge0}$ is $\mathcal{L}_{p,\phi}$-bounded, then $f \in \mathcal{L}_{p,\phi}$ and

$$\|f\|_{\mathcal{L}_{p,\phi}} \le \sup_{n\ge0} \|f_n\|_{\mathcal{L}_{p,\phi}}.$$

(ii) *If* $f \in \mathcal{L}^{\natural}_{p,\phi}$*, then* $(f_n)_{n\geq 0}$ *is* $\mathcal{L}^{\natural}_{p,\phi}$*-bounded and*

$$\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \geq \sup_{n\geq 0} \|f_n\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

Conversely, if $(f_n)_{n\geq 0}$ *is* $\mathcal{L}^{\natural}_{p,\phi}$*-bounded, then* $f \in \mathcal{L}^{\natural}_{p,\phi}$ *and*

$$\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \leq \sup_{n\geq 0} \|f_n\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

Remark 4.1. In general, for $f \in \mathcal{L}_{p,\phi} \cap L_{1,0}$ (res. $f \in \mathcal{L}^{\natural}_{p,\phi}$), its corresponding martingale $(f_n)_{n\geq 0}$ with $f_n = E_n f$ does not always converge to f in $\mathcal{L}_{p,\phi}$ (res. $\mathcal{L}^{\natural}_{p,\phi}$). See Remark 3.7 in [9] for the case $\phi(r) = r^{\lambda}$.

Proof of Theorem 1.4. Let $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi})$ and $f \in \mathcal{L}^{\natural}_{p,\phi}(\mathcal{F}_n)$. Then, using Proposition 4.1, we have

$$\|E_n[g]f\|_{\mathcal{L}^{\natural}_{p,\phi}} = \|E_n[gf]\|_{\mathcal{L}^{\natural}_{p,\phi}} \leq \|gf\|_{\mathcal{L}^{\natural}_{p,\phi}} \leq \|g\|_{Op}\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

Therefore, we have $E_n g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}(\mathcal{F}_n))$.

Conversely, assume that $E_n g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi}(\mathcal{F}_n))$ and $\sup_{n\geq 0} \|E_n g\|_{Op} < \infty$. Then, using Proposition 4.1 and Theorem 1.1, we have

$$\|g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \|g\|_{L_\infty} \leq \sup_{n\geq 0} \|E_n g\|_{\mathcal{L}_{p,\phi/\phi_*}} + \sup_{n\geq 0} \|E_n g\|_{L_\infty} \lesssim \sup_{n\geq 0} \|E_n g\|_{Op} < \infty.$$

Using Theorem 1.1 again, we have $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p,\phi})$. □

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